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A mathematical model for the density-augmented spreading of a particle-laden discharge in a turbulent shallow-water flow

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Abstract. A two-dimensional horizontal dispersion equation is derived for the many ways that buoyancy modifies the spreading of a mono-disperse discharge of particles in a shallow-water flow. The particle-induced lateral density gradient gives rise to a secondary transverse flow which tends to carry particles outwards. Stratification with the consequent reduced mixing, changes the local vertical profile of the original flow and gives rise to a Burgers nonlinearity (slowing down of sinking particles or a speeding up of buoyant particles). Stratification also modifies the non-local horizontal distribution of the current, with an inflow towards particle-laden regions where the drag is reduced. A simple eddy-diffusivity turbulence model is used which permits explicit evaluation of most of the linear and nonlinear coefficients. Nonlinear effects are shown to be significant for particle volume concentrations of only 10 parts per million.

1. Introduction

The dumping of waste materials in shallow water flows often involves particulate matter (eg sewage sludge). The initial slumping of an abrupt discharge is dominated by the density [1]. Any suspended material is transported by the buoyancy-modified flow, gradually disperses horizontally and becomes more dilute. Finally the dispersion process will become linear (although the asymptotic horizontal velocity and horizontal shear diffusivity for sinking particles [2], [3] are different from those for solutes). The longest-lasting and largest length-scale effects of density will arise just before that final stage.

Some of the ways in which the density change associated with particles can modify the flow are illustrated in Figs. 1a-c. In the original flow direction (Fig. 1a) the velocity profile can become steepened where the turbulent mixing is reduced [4]. This corresponds to a slight slowing down of the effective longitudinal velocity of sinking particles and a speeding-up of rising particles. In the cross-flow direction (Fig. 1b) dense sinking particles would induce a secondary circulation which tends to carry particles outwards. For buoyant rising particles the circulation would be reversed but the particles would again tend to be carried outwards. In the horizontal plane (Fig. 1c) there would be an inflow towards particle-laden regions where, as a consequence of the reduced turbulent mixing, there is reduced drag and the flow is speeded up.

Even with the restriction to the immediate pre-linear stage, there is a wide range of possible physical regimes. For example, there could be different modes of discharge (abrupt/steady/intermittent/point/distributed) and different total weight or flux of particles. In order to incorporate as wide a range of circumstances as possible into a single calculation scheme, the method of maximum-generality scalings is used [5], [6]. A physical regime is identified in which the geometrical, flow and discharge parameters are such that as many mechanisms as possible are represented in the final approach to the linear dispersion regime. An actual flow

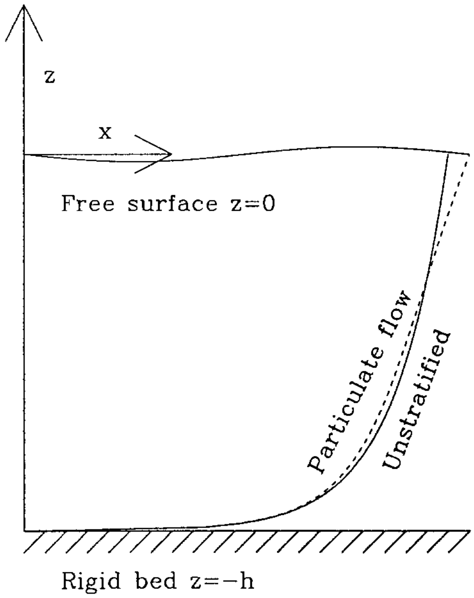

Fig. 1a. Stratification changes the velocity profile.

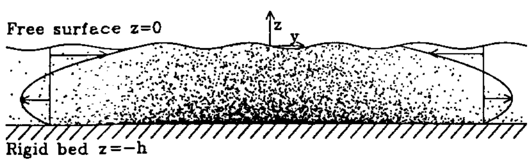

Fig. 1b. Particles and the density-driven cross flow.

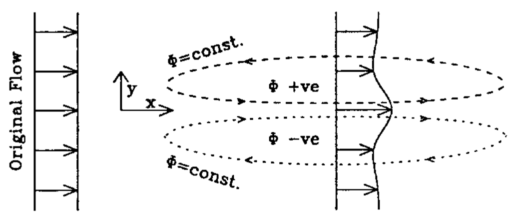

Fig. 1c. Local drag reduction changes the horizontal flow.

may not have these scales and the analysis could then be simplified by the deletion of some terms.

For a uni-directional flow in the x-direction in water of depth h with an eddy-viscosity turbulence model, the vertically averaged particle volume fraction $\langle c\rangle$ for mono-disperse particles is shown to satisfy an equation of the form

$$\partial_t(h\langle c\rangle) + \partial_x\left(\left[hu_0 + hu_1\langle c\rangle + \frac{u_0}{\langle u\rangle}\,\partial_y\Phi\right]\langle c\rangle\right) - \partial_y\left(\frac{u_0}{\langle u\rangle}\,\partial_x\Phi\langle c\rangle\right) - \partial_x(hE_0\,\partial_x\langle c\rangle) - \partial_y(h[K_0 + K_1\langle c\rangle]\partial_y\langle c\rangle) = 0\,. \tag{1.1}$$

Here u_0, E_0 and K_0 are the effective horizontal speed, longitudinal shear dispersion and transverse turbulent eddy diffusivity for dilute particles [2], [3]. The vertical re-distribution of the flow and of the particles changes the effective longitudinal transport of particles and is quantified via the Burgers nonlinearity u_1. The horizontal re-distribution of the flow is accounted for in the horizontal stream function Φ. The nonlinear transverse diffusion, with coefficient K_1, is associated with the lateral outflow driven by the lateral density gradient (see Fig. 1b).

For solutes there is little vertical density variation and all the nonlinear coefficients u_1, K_1, Φ are zero. Instead, the final effects of buoyancy arise at higher order [5], [6]. Hence, the effects of buoyancy and nonlinearity are more pronounced for particulate discharges than for solutes. Indeed, an illustrative example reveals that dense particles with depth average volume concentrations of 10 parts per million can exhibit significant nonlinearity.

This work is a preliminary contribution to a multi-disciplinary study of fine particle transport (RACS–350) commencing April 1995, involving the Universities of Birmingham, Bradford, Loughborough and Sheffield together with the IMER laboratory at Plymouth. So, it remains to be determined whether the versatility sought with equation (1.1) will be refuted or confirmed by computer simulations, laboratory experiments and field observations.

2. Full equations

Even for a constant depth channel with neither erosion or deposition and with an eddy-diffusivity model for the turbulence the coupled equations for the particle volume fraction $c(x, y, z, t)$ and the velocity components (u, v, w) are formidably complicated:

$$\partial_t c + u\partial_x c + v\partial_y c + (W + w)\partial_z c = \partial_x(\kappa_1\partial_x c) + \partial_y(\kappa_2\partial_y c) + \partial_z(\kappa_3\partial_z c)\,, \tag{2.1a}$$

$$\begin{aligned}&\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u + \partial_x p - G\\ &\quad = \partial_x\{2\nu_{11}\partial_x u\} + \partial_y\{\nu_{12}(\partial_y u + \partial_x v)\} + \partial_z\{\nu_{13}(\partial_z u + \partial_x w)\}\,,\end{aligned} \tag{2.1b}$$

$$\begin{aligned}&\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + \partial_y p\\ &\quad = \partial_x\{\nu_{12}(\partial_y u + \partial_x v)\} + \partial_y\{2\nu_{22}\partial_y v\} + \partial_x\{\nu_{23}(\partial_z v + \partial_y w)\}\,,\end{aligned} \tag{2.1c}$$

$$\begin{aligned}&\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w + \partial_z p - \alpha g c\\ &\quad = \partial_x\{\nu_{13}(\partial_z u + \partial_x w)\} + \partial_y\{\nu_{23}(\partial_z v + \partial_y w)\} + \partial_z\{2\nu_{33}\partial_z w\}\,,\end{aligned} \tag{2.1d}$$

$$\partial_x u + \partial_y v + \partial_z w = 0\,, \tag{2.1e}$$

$$W_c - \kappa_3\partial_z c = u = v = w = 0 \text{ on } z = -h\,, \tag{2.1f}$$

$$W_c - \kappa_3\partial_z c = \nu_{13}\partial_z u = \nu_{23}\partial_z v = w = 0 \text{ on } z = 0\,, \tag{2.1g}$$

with

$$\kappa_i = \kappa_i[Ri]\,, \quad \nu_{ij} = \nu_{ij}[Ri]\,, \tag{2.1h}$$

where

$$Ri = \alpha g\partial_z c/\{(\partial_z u)^2 + (\partial_z v)^2\}\,. \tag{2.1i}$$

Here W is the vertical rise velocity of the mono-disperse particles (negative for sinking particles), κ_i are the eddy diffusivities for concentration (with principal axes in the coordinate directions), ν_{ij} are the eddy diffusivities for momentum, $-G$ is the pressure gradient which drives the basic uni-directional flow, p the particle-related pressure perturbation, αg the upwards acceleration (negative for sinking particles), h the constant water depth and Ri the gradient Richardson number. The boundary conditions (2.1f, g) do not allow the particles to settle from the flow. The square brackets in equations (21.1h) allows for the possibility that the eddy diffusivities κ_i, ν_{ij} could be functionals of the Richardson number (i.e. stratification at one level could modify the turbulence experienced at the other levels in the flow). Except for the inclusion of the vertical rise velocity W, the above equations (2.1) are the same as the equations (4) studied in the context of solutes by Smith [6]. A higher-order turbulence closure model, such as a turbulent kinetic energy-dissipation model, would require further equations for the turbulent kinetic energy and the dissipation instead of the Richardson number relationship (2.1h).

3. Choice of scalings

To render the equations mathematically tractable we assume that the depth-to-width ratio δ and the friction-to-bulk velocity ratio ε:

$$\delta = h/B \quad \text{and} \quad \varepsilon = u_*/U \tag{3.1}$$

are both small but non-zero. Using the depth h and the undisturbed bulk flow velocity U as basic dimensional quantities, we seek to specify the sizes of the profusion of terms in equations (2.1). In particular, we characterise the eddy diffusivities as being of size

$$\kappa_i, \nu_{ij} \sim hU\varepsilon \tag{3.2}$$

(i.e. scaling as the depth times friction velocity). An immediate consequence is that the time scale of the lateral diffusion process is of order $\delta^{-2}\varepsilon^{-1}h/U$.

The idea of maximum generality scalings is to determine a relationship between the (independent) small parameters δ and ε (and any other parameters) so that when we simplify equation (2.1) by invoking the smallness of the aspect ratio δ, we still retain as many terms and as much of the physics as possible. An actual flow may not have the requisite relationship between δ and ε so some unnecessarily small terms may have been retained but no large terms have been overlooked. Here the new feature is the vertical rise (or sinking) velocity W of the particles. If the particles are to be distributed non-uniformly with respect to z then we can deduce from the particle diffusion equation (2.1a) that W must be of the size

$$W \sim U\varepsilon\,. \tag{3.3}$$

This is the first step of the lengthy argument symbolised in Fig. 2. For simplicity we shall ignore detailed complications such as the size of the eddy diffusivities becoming smaller close to the bed.

In the lateral y-direction the basic length scaled is $B \sim h\delta^{-1}$. If the lateral distribution of particles is to be influenced both by diffusion κ_2 and by advection with the secondary flow v, then we can infer that the lateral velocity v has the size

$$v \sim U\delta\varepsilon\,. \tag{3.4}$$

We remark that this estimate (3.4) is much smaller than the corresponding estimate $U\varepsilon$ for solutes [6]. By virtue of mass conservation of water the secondary flow has zero vertical average value. This still allows a net lateral transport of the vertically non-uniform particles. However, at the first approximation solutes have the same uniform vertical distribution as the water itself. So, for a given secondary flow the transport of a solute is order δ smaller than the transport of particles. Equivalently, to achieve the same transport of solutes the secondary flow needs to be larger by a factor δ^{-1}.

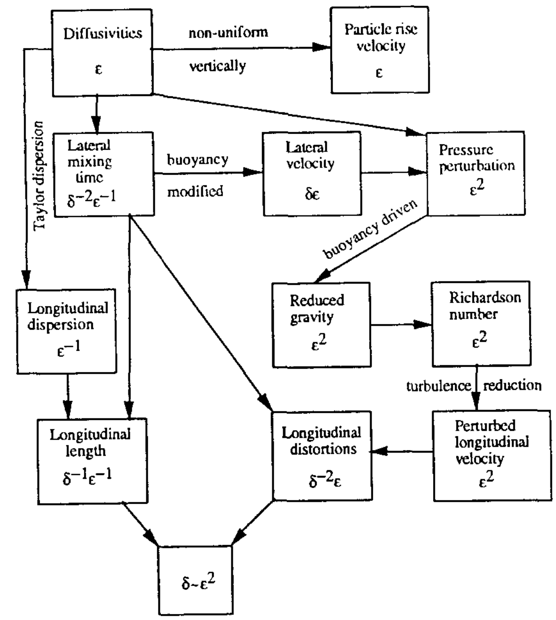

Fig. 2. Derivation of scalings to incorporate as many physical effects as possible in the dispersion equation.

To drive the secondary flow the particle-related pressure perturbation p must have the size

$$p \sim U^2 \varepsilon^2 . \tag{3.5}$$

For this pressure to be associated with a particle volume concentration of order unity (with respect to appropriate concentration units e.g. parts per million) the buoyancy acceleration αg would have to be of the size

$$\alpha g \sim \frac{U^2}{h} \varepsilon^2 . \tag{3.6}$$

The corresponding estimate for the order of magnitude of the gradient Richardson number is

$$Ri \sim \varepsilon^2 . \tag{3.7}$$

If we assume that turbulence suppression is linearly proportional to Ri, then there will be a perturbed longitudinal velocity u' of size

$$u' \sim U \varepsilon^2 . \tag{3.8}$$

On the time scale $\delta^{-2}\varepsilon^{-1}h/U$ of the lateral diffusion, the perturbed longitudinal velocity will cause longitudinal distortions of size

$$L \sim h \delta^{-2} \varepsilon . \tag{3.9}$$

In the longitudinal direction the shear dispersion coefficient E_0 depends inversely upon the vertical diffusivity κ_3 [7]. Hence, we can estimate

$$E_0 \sim hU \varepsilon^{-1} . \tag{3.10}$$

On the time scale of the lateral diffusion the corresponding longitudinal diffusive length scale is of size

$$L \sim h\delta^{-1}\varepsilon^{-1} . \tag{3.11}$$

Compatibility between the two estimates (3.9, 3.11) of the longitudinal length scale leads us to consider the physical regime in which δ and ε have sizes such that

$$\delta \sim \varepsilon^2 . \tag{3.12}$$

The significance of this maximum-generality selection of the scalings, is that in this special circumstance *all* the above mentioned physical mechanisms are of comparable importance (transverse diffusion, vertical rise, secondary flow, turbulence reduction, longitudinal shear dispersion). It happens that the $\delta \sim \varepsilon^2$ relationship value is the same as that deduced by Smith [6] for solutes. However, the scalings for other quantities are not the same as for solutes. In particular the secondary flow and the reduced buoyancy are both a factor ε^2 smaller for particles than for solutes (i.e. nonlinearity and buoyancy are more pronounced for sinking particles than for solutes).

As an illustrative example, appropriate to a shallow river, we specify the basic physical parameters

$$U = 0.1\text{ms}^{-1}, \quad h = 2\text{m}, \quad B = 50\text{m} \quad \text{i.e.} \quad \delta \sim \frac{1}{25}, \varepsilon \sim \frac{1}{5} . \tag{3.13}$$

The implied scales for the other major physical parameters are

$$\begin{aligned} &\text{eddy diffusivities} \sim 0.04\text{m}^2 s^{-1}, \text{rise velocity} \sim 0.02\text{m}s^{-1}, \text{length} \sim 250\text{m}, \\ &\text{total excess (or deficit) weight of particles in the flow} \sim 0.5 \text{ tonnes}, \end{aligned} \tag{3.14}$$

The particle volume fraction would be about 20 parts per million when the width and length scale correspond to this most complicated regime.

These scalings do represent a physically realistic possibility. So, the maximum-generality scalings are not just a mathematical idealisation (requiring a massive discharge or strange physical constants). For a discharge with different total excess (or deficit) weight of particles or in a regime with different depth, length and breadth scales (or when we account for the eddy diffusivities becoming small close to the bed), some of the retained terms in the maximum-generality scalings will be relatively small. The crucial feature of the scalings is that no important terms are neglected in the weakly nonlinear pre-asymptotic regime.

4. Scaled equations

Instead of continuing with the two small parameters δ and ε, we now assume the relationship (3.12) and eliminate ε in favour of δ. In axes moving at the (as yet undetermined) effective horizontal velocity u_0 and with the relative sizes of terms made explicit, the full equations (2.1) can be re-written

$$\begin{aligned} &\delta^2\partial_t c + \delta(u - u_0)\partial_x c + \delta^2 v\partial_y c + (W + \delta^2 w)\partial_z c \\ &\quad = \partial_z(\kappa_3\partial_z c) + \delta^2\partial_y(\kappa_2\partial_y c) + O(\delta^3), \end{aligned} \tag{4.1a}$$

$$\delta(u - u_0)\partial_x u - G = \partial_z(\nu_{13}\partial_z u) + O(\delta^2), \tag{4.1b}$$

$$\partial_y p = \partial_z(\nu_{23}\partial_z v) + O(\delta), \tag{4.1c}$$

$$\partial_z p - \alpha g c = O(\delta)\,, \tag{4.1d}$$
$$\partial_x u + \delta\partial_y v + \delta\partial_z w = O(\delta^2)\,, \tag{4.1e}$$
$$Wc - \kappa_3\partial_z c = u = v = w = 0 \text{ on } z = -h\,, \tag{4.1f}$$
$$Wc - \kappa_3\partial_z c = \nu_{13}\partial_z u = \nu_{23}\partial_z v = w = 0 \text{ on } z = 0\,, \tag{4.1g}$$
$$\text{with } \kappa_i = \kappa[Ri]\,, \quad \nu_{ij} = \nu_{ij}[Ri]\,, \tag{4.1h}$$
$$\text{where } Ri = \delta\frac{\alpha g \partial_z c}{(\partial_z u)^2} + O(\delta^2)\,. \tag{4.1i}$$

By construction these equations retain all the nonlinear interactions discussed in the previous section. Yet already, these equations (4.1) are far less formidable than the original equations (2.1). To avoid the additional formal mathematical step of making all dependent and independent variables non-dimensional, the equations (4.1) have been written in a dimensional but scaled form. Dimensional unscaled results can be recovered by setting $\delta = 1$.

To exploit still further the smallness of the parameter δ we represent the flow quantities by regular power series in δ:

$$\text{e.g.}\quad \kappa_3 = \kappa_3^{(0)} + \delta\kappa_3^{(1)} + \cdots\,, \tag{4.2}$$

where all the $\kappa_3^{(j)}$ are independent of δ. In particular, to the first approximation the eddy diffusivities are unaffected by stratification and the vertical concentration profile has the equilibrium shape

$$\gamma(z) = \exp\left(W\int_{z_0}^{z}\frac{dz'}{\kappa_3^{(0)}}\right)\,. \tag{4.3a}$$

A convenient choice for the reference level z_0 is to ensure that γ has unit vertical average value:

$$\langle\gamma\rangle = 1 \quad \text{where} \quad \langle\gamma\rangle = \frac{1}{h}\int_{-h}^{0}\gamma\, dz\,. \tag{4.3b,c}$$

It is the vertical non-uniformity of this equilibrium concentration profile γ that gives rise to the different physics and mathematics for the dispersion of mono-disperse particles as compared with the dispersion of buoyant solutes. For dilute suspensions Giddings [8] showed that the mathematical differences could be reduced considerably if the γ profile were factored out:

$$c(x, y, z, t; \delta) = b(x, y, z, t; \delta)\gamma(z)\,. \tag{4.4}$$

From equation (4.1a) we can deduce that b satisfies the equation

$$\begin{aligned}&\delta^2\gamma\partial_t b + \delta(u - u_0)\gamma\partial_x b + \delta^2 v\gamma\partial_y b + \delta^2 w\partial_z(\gamma b)\\ &\quad = \partial_z(\kappa_3\gamma\partial_z b) + \partial_z((\kappa_3 - \kappa_3^{(0)})b\partial_z\gamma) + \delta^2\partial_y(\gamma\kappa_2\partial_y b) + O(\delta^3)\,,\end{aligned} \tag{4.5a}$$

with

$$\kappa_3\gamma\partial_z b + \left\{\kappa_3 - \kappa_3^{(0)}\right\}\partial_z\gamma = 0 \quad \text{on} \quad z = -h, 0\,. \tag{4.5b}$$

At a first approximation a regular series expansion for b yields the result

$$b^{(0)} = \langle c\rangle\,. \tag{4.6}$$

Hence, the weighted-concentration $b^{(0)}$ is independent of the vertical coordinate z. Also, the normalisation (4.3b) permits us to identify this uniform value with the vertically averaged concentration $\langle c\rangle$.

5. Flow field

For the longitudinal velocity the leading-order solution is unaffected by the presence of particles

$$u^{(0)} = G^{(0)} \int_{-h}^{z} \frac{(-z')}{\nu_{13}^{(0)}}\, \mathrm{d}z' , \tag{5.1a}$$

where the pressure gradient $G^{(0)}$ is determined from the bulk flow:

$$G^{(0)} \int_{-h}^{0} \frac{z^2}{\nu_{13}^{(0)}}\, \mathrm{d}z = h\langle u^{(0)}\rangle . \tag{5.1b}$$

For simplicity we shall delay investigation of the sensitivity of these integrals to any near-bed region in which $\nu_{13}^{(0)}$ becomes anomalously small. The stratification is weak with Richardson number of order δ:

$$Ri^{(1)} = \langle c\rangle \alpha g \partial_z \gamma / (\partial_z u^{(0)})^2 . \tag{5.2}$$

The associated reduction in eddy viscosity will likewise be of order δ and be proportional to the vertically averaged particle concentration $\langle c\rangle$. Any change in drag (as quantified by the pressure gradient) may also modify the eddy viscosity. Hence, we represent the fractional change in eddy viscosity by the sum of two terms:

$$\nu_{13}^{(1)}/\nu_{13}^{(0)} = -\langle c\rangle R(z) + \frac{1}{2} G^{(1)}/G^{(0)}. \tag{5.3a}$$

It is in the specification of the shape function $R(z)$ that we need a turbulence model.

The Monin-Obukhov [4] turbulence model used later in this paper gives the function $R(z)$ as a weighted integral of the perturbation Richardson number $Ri^{(1)}$:

$$\langle c\rangle R(z) = \frac{5}{h} \int_{-h}^{z} \left(\frac{-z'}{h+z'}\right) Ri^{(1)}\, \mathrm{d}z' . \tag{5.3b}$$

The usual turbulence scaling of the eddy viscosity with the friction velocity U_* (i.e. the square-root of the bed stress) implies that the coefficient of $G^{(1)}/G^{(0)}$ in equation (5.3a) has the stated value of a half.

The first-order longitudinal velocity correction $u^{(1)}$ comprises contributions associated with the reduction in eddy viscosity and the change in pressure gradient

$$u^{(1)} = \langle c\rangle \int_{-h}^{z} R \partial_z u^{(0)}\, \mathrm{d}z + \frac{G^{(1)}}{2G^{(0)}} u^{(0)} . \tag{5.4}$$

The value of $G^{(1)}/G^{(0)}$ can be determined from the physical constraint that when integrated across the entire flow, there is zero net flow of water associated with the presence of particles

$$\frac{G^{(1)}}{2G^{(0)}} \langle u^{(0)}\rangle = -\bar{c}\mathfrak{U} \quad \text{with} \quad \mathfrak{U} = \int_{-h}^{0} \left(\frac{-z}{h}\right) R \partial_z u^{(0)}\, \mathrm{d}z . \tag{5.5a,b}$$

Here (in the context of constant water depth) $\tilde{c}$ denotes the cross-sectionally averaged concentration. In the centre of the particle-laden water, where $\langle c\rangle$ exceeds $\tilde{c}$, the turbulence reduction is associated with a slight speeding up of the flow. At the sides there is a compensating return flow.

At order δ the vertically averaged form of the mass conservation equation (4.1e) is

$$\partial_x\langle u^{(1)}\rangle + \partial_y\langle v^{(0)}\rangle = 0 . \tag{5.6}$$

In terms of the above solution (5.4) for $u^{(1)}$ we can define a horizontal stream function

$$\Psi = \mathfrak{U}\int_{y-}^{y} (\langle c\rangle - \tilde{c})\,\mathrm{d}y' , \tag{5.7a}$$

$$\langle v^{(0)}\rangle = -\partial_x\Psi , \tag{5.7b}$$

where y_- is the boundary (possibly at minus infinity) of the constant depth region.

In the vertical direction the pressure perturbation is hydrostatic

$$p^{(0)} = \langle p^{(0)}\rangle = \alpha g\langle c\rangle\left\{\langle (h+z)\gamma\rangle - \int_z^0 \gamma\,\mathrm{d}z'\right\} . \tag{5.8}$$

Here the term $\langle (h+z)\gamma\rangle$ is the height above the bed of the centroid for the particle concentration profile.

The two contributions to the pressure perturbation $p^{(0)}$ give rise to a corresponding two contributions to the lateral velocity $v^{(0)}$:

$$v^{(0)} = -\frac{\partial_y\langle p^{(0)}\rangle}{G^{(0)}}V(z) - \frac{\alpha g h\partial_y\langle c\rangle}{G^{(0)}}B(z) , \tag{5.9a}$$

where

$$V(z) = G^{(0)}\int_{-z}^{z}\frac{(-z')}{\nu_{23}^{(0)}}\,\mathrm{d}z' \tag{5.9b}$$

and

$$B(z) = \frac{G^{(0)}}{h}\int_{-h}^{z}\frac{1}{\nu_{23}^{(0)}}\int_{z'}^{0}\left\{\langle (h+z)\gamma\rangle - \int_{z''}^{0}\gamma\,\mathrm{d}z'''\right\}\mathrm{d}z''\,\mathrm{d}z' . \tag{5.9c}$$

The presence of the longitudinal pressure gradient $G^{(0)}$ in the definition (5.9b) of $V(z)$ ensures that if the viscosities $\nu_{13}^{(0)}$ and $\nu_{23}^{(0)}$ are equal then so are the velocities $u^{(0)}(z)$ and $V(z)$.

We can re-write the expression (5.9a) for $v^{(0)}$:

$$v^{(0)} = -\frac{\partial_y\langle p^{(0)}\rangle + \alpha g h\partial_y\langle c\rangle\xi}{G^{(0)}}V(z) - \frac{\alpha g h\partial_y\langle c\rangle}{G^{(0)}}(B(z) - \xi V(z)), \tag{5.10a}$$

where

$$\xi = \int_{-h}^{0}\frac{(-z)}{\nu_{23}^{(0)}}\int_{z}^{0}\left\{\langle (h+z)\gamma\rangle - \int_{z'}^{0}\gamma\,\mathrm{d}z''\right\}\mathrm{d}z'\,\mathrm{d}z \Big/ h\int_{-h}^{0}\frac{z^2}{\nu_{23}^{(0)}}\,\mathrm{d}z . \tag{5.10b}$$

and the combination $B(z) - \xi V(z)$ has been constructed to have zero vertical average. Thus the second term in (5.10a) can be identified as the buoyancy-driven transverse flow (Fig. 1b) and the first term can be identified as the mean flow (5.7b) induced by drag reduction, with

$$\frac{\partial_x \Psi}{\langle V \rangle} = -\frac{\partial_y \langle p^{(0)} \rangle + \alpha g h \partial_y \langle c \rangle \xi}{G^{(0)}} . \tag{5.10c}$$

If the viscosities $\nu_{13}^{(0)}$ and $\nu_{23}^{(0)}$ are equal, the definition (5.10b) for the parameter ξ can be re-written

$$\xi = \frac{1}{h^2 \langle u^{(0)} \rangle} \int_{-h}^{0} \left(u^{(0)} - \langle u^{(0)} \rangle \right) \left\{ \langle (h+z)\gamma \rangle - \int_z^0 \gamma \, dz' \right\} d . \tag{5.11}$$

For future reference we note that if $\nu_{13}^{(0)}$ becomes anomalously small near the bed, then the velocity shear $(u^{(0)} - \langle u^{(0)} \rangle)$ is small relative to the bulk velocity $\langle u^{(0)} \rangle$ and the parameter ξ will be anomalously small.

The above formulae (5.4–5.11) enable us to determine the particle-induced change to the flow field. Our next task is to calculate the consequences as regards the horizontal evolution of the particle distribution $\langle c \rangle (x, y, t)$.

6. Shear dispersion

If we regard the flow field u, v, w and eddy diffusivities $\kappa_1, \kappa_2, \kappa_3$ as being known, then the solution of the equations (4.6a, b) for $b(x, y, z, t; \delta)$ is similar to that for a passive solute [7], [9]. Already, we have made substantial use of the fact that at the lowest approximation $b^{(0)}$ is independent of the vertical coordinate z and can be identified with the vertically averaged particle concentration $\langle c \rangle$.

The order δ terms in equations (4.5a, b) yield a boundary value problem for the perturbation $b^{(1)}$:

$$\partial_z \left(\gamma \kappa_3^{(0)} \partial_z b^{(1)} \right) = \gamma \left(u^{(0)} - u_0 \right) \partial_x \langle c \rangle - \langle c \rangle \partial_z \left(\kappa_3^{(1)} \partial_z \gamma \right) , \tag{6.1a}$$

with

$$\gamma \kappa_3^{(0)} \partial_z b^{(1)} = -\langle c \rangle \kappa_3^{(1)} \partial_z \gamma \quad \text{on} \quad z = -h, \, 0. \tag{6.1b}$$

If we integrate the differential equation (6.1a) from the bed to the free surface, then we can deduce that the previously undetermined axes velocity u_0 must have the value

$$u_0 = \langle \gamma u^{(0)} \rangle . \tag{6.2}$$

Except for notational differences, this is equivalent to the result obtained by Binnie and Phillips [2] and extended by Elder [3]. A first integration of equation (6.1a) then yields the expression

$$\gamma \kappa_3^{(0)} \partial_z b^{(1)} = -\partial_x \langle c \rangle \int_z^0 \gamma (u^{(0)} - u_0) \, dz' - \langle c \rangle \kappa_3^{(1)} \partial_z \gamma . \tag{6.3}$$

If we invoke the Reynolds analogy and take $\kappa_3^{(j)}$ to be proportional to $\nu_{i3}^{(j)}$, then using the turbulence reduction model (5.3a) we can rewrite the expression (6.3)

$$\partial_z b^{(1)} = \frac{\partial_x \langle c \rangle}{\gamma \kappa_3^{(0)}} \int_z^0 \gamma (u^{(0)} - u_0) \, dz' + \langle c \rangle \frac{\partial_z \gamma}{\gamma} \left(\langle c \rangle R + \tilde{c} \frac{\mathfrak{U}}{\langle u^{(0)} \rangle} \right) . \tag{6.4}$$

The vertical average of the order δ^2 terms in equation (4.5a) gives us the time-dependence of $\langle c\rangle$:

$$\partial_t\langle c\rangle + \partial_x\langle\gamma(u^{(0)}-u_0)b^{(1)}\rangle + \partial_x(\langle c\rangle\langle\gamma u^{(1)}\rangle) + \partial_y(\langle c\rangle\langle\gamma v^{(0)}\rangle) = \partial_y(\langle\gamma\kappa_2^{(0)}\rangle\partial_y\langle c\rangle)\,. \tag{6.5}$$

If we use the formulae (6.4, 5.4, 5.10a) to replace $\partial_z b^{(1)}, u^{(1)}$ and $v^{(0)}$ in terms of $\langle c\rangle$, then we can derive the two-dimensional weakly nonlinear shear dispersion equation

$$\begin{aligned}\partial_t\langle c\rangle &+ \partial_x\left(u_1\langle c\rangle^2 + u_2\tilde{c}\langle c\rangle + \frac{\langle\gamma u^{(0)}\rangle}{\langle u^{(0)}\rangle}\partial_y\Psi\langle c\rangle\right) - \partial_y\left(\frac{\langle\gamma V\rangle}{\langle V\rangle}\partial_x\Psi\langle c\rangle\right)\\ &= E_0\partial_x^2\langle c\rangle + \partial_y\{(K_0 + K_1\langle c\rangle)\partial_y\langle c\rangle\}\,,\end{aligned} \tag{6.6a}$$

with

$$u_1 = -\frac{1}{h}\int_{-h}^{0} R\frac{\partial_z\gamma}{\gamma}\int_{-h}^{z}\gamma(u^{(0)}-u_0)\,dz'\,dz - \frac{1}{h}\int_{-h}^{0} R\partial_z u\int_{-h}^{z}(\gamma-1)\,dz'\,dz\,, \tag{6.6b}$$

$$u_2 = \frac{-\mathfrak{U}}{\langle u^{(0)}\rangle h}\int_{-h}^{0}\frac{\partial_z\gamma}{\gamma}\int_{-h}^{z}(u^{(0)}-u_0)\,dz'\,dz\,, \tag{6.6c}$$

$$E_0 = \frac{1}{h}\int_{-h}^{0}\frac{1}{\gamma\kappa_3^{(0)}}\left\{\int_z^0\gamma(u^{(0)}-u_0)\,dz'\right\}^2 dz\,, \tag{6.6d}$$

$$K_0 = \langle\gamma\kappa_2^{(0)}\rangle\,, \tag{6.6e}$$

$$\begin{aligned}K_1 = \alpha g\Bigg\{&\frac{1}{h}\int_{-h}^{0}\frac{1}{\nu_{23}^{(0)}}\left(\int_z^0(\gamma-1)\,dz'\right)\left(\int_z^0\langle(h+z)\gamma\rangle - \int_{z'}^0\gamma\,dz'\right)dz\\ &-\xi\int_{-h}^{0}\frac{(-z)}{\nu_{23}^{(0)}}\int_z^0(\gamma-1)\,dz'\,dz\Bigg\}\,.\end{aligned} \tag{6.6f}$$

In the absence of vertical drift $\gamma = 1$ and all the nonlinear coefficients u_1, u_2, Ψ, K_1 are zero. For future reference we note that if the velocity shear $(u^{(0)} - u_0)$ is small relative to the bulk velocity $\langle u^{(0)}\rangle$, then the term u_2 will be small relative to u_1. Also, E_0 will be smaller than anticipated.

This equation (6.6a) is the key result of this paper. The equation (1.1) stated in the introduction neglects u_2 and involves the original stationary axes without any δ-scalings. Also, in equation (1.1) the restriction to constant water depth h has been relaxed and the horizontal stream function Ψ has been replaced by

$$\Phi = h\Psi = \int_{y_-}^{y} h\mathfrak{U}(\langle c\rangle - \tilde{c})\,dy'\,, \tag{6.7a}$$

with

$$\tilde{c} = \int_{y_-}^{y_+} h\mathfrak{U}\langle c\rangle\,dy' \Big/ \int_{y_-}^{y_+} h\mathfrak{U}\,dy'\,. \tag{6.7b}$$

If the particles are well-mixed across the entire flow then $\tilde{c} = \langle c \rangle$ and equation (6.6a) reduces to the well-known Burgers [10] equation

$$\partial_t \langle c \rangle + (u_1 + u_2)\partial_x(\langle c \rangle^2) = E_0 \partial_x^2 \langle c \rangle \,. \tag{6.8}$$

The solutions are nonlinear distortions of the solutions of the linear diffusion equation [11], [12].

7. Coefficients for the linear theory

The above formulae (6.2, 6.6c, d) for u_0, E_0, and K_0 involve a profusion of vertical average values. Following Elder [3], Sumer [13] and Smith [14] we use a parabolic model for the vertical eddy diffusivity

$$\kappa_3^{(0)} = \nu_{13}^{(0)} = \nu_{23}^{(0)} = k u_* h (1-\eta)\eta \quad \text{with} \quad \eta = \eta_* + \left(1 + \frac{z}{h}\right)(1 - \eta_*)\,, \tag{7.1a,b}$$

where k is von Kármán's constant (about 0.4), u_* the friction velocity h the water depth and η_* the dimensionless roughness height (0.001). The classical logarithmic solution for the undisturbed velocity profile is

$$u^{(0)} = \langle u \rangle + \frac{u_*}{k}[1 + \ln \eta] \quad \text{where} \quad u_* = \frac{k\langle u \rangle}{\left[\ln\left(\frac{1}{\eta_*}\right) - 1\right]}\,. \tag{7.2a,b}$$

The relationship (7.2b) between u_* and η_* ensures that the velocity $u^{(0)}$ is zero at $\eta = \eta_*$.

The boundary layer character of the velocity profile (7.2a) implies that, except very close to the bed, the velocity $(u^{(0)} - \langle u \rangle)$ is small relative to the bulk velocity $\langle u \rangle$. The robust nature of the maximum generality scalings is such that it is not necessary to repeat the analysis. Though, as anticipated earlier, two of the nonlinear terms ξ and u_2 become smaller than estimated and could have been neglected.

To quantify the effect of the vertical drift velocity W upon the equilibrium particle concentration profile $\gamma(z)$, we use the vertical Péclet number

$$P = \frac{W}{k u_*}\,. \tag{7.3}$$

If we ignore terms of order η_*, then the profile γ has the power law solution (see Fig. 3)

$$\gamma = \frac{\sin \pi P}{\pi P}\left[\frac{\eta}{1-\eta}\right]^P \quad -1 < P < 1\,. \tag{7.4}$$

For particles with rise velocity greater than ku_* the turbulence is too weak to detach the strongly buoyant particles from the free surface. Similarly, for dense particles with fall velocity greater than ku_*, the particles cannot be detached from the bed.

For the effective velocity $\langle \gamma u^{(0)} \rangle$ of the particles, Sumer [13], equation (35) gives a formula involving the psi function. Fig. 4 gives a graph (continuous curve) of the exact result and of the ad hoc approximation (dashed curve)

$$(u_0 - \langle u \rangle)k/u_* = \frac{P}{2(1+P}\left(\frac{\pi^2}{3} + P - \left[\frac{\pi^2}{3} - 3\right]P^2\right)\,. \tag{7.5}$$

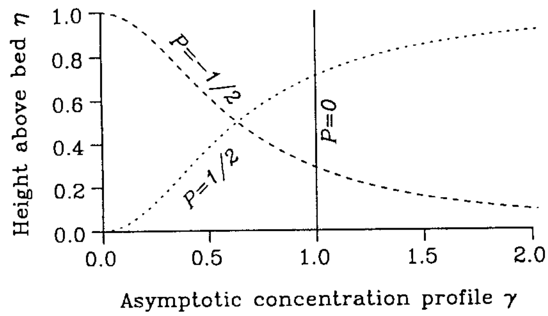

Fig. 3. Illustration of the vertical concentration profiles for rising ($P = 1/2$), neutral ($P = 0$) and sinking ($P = -1/2$) particles.

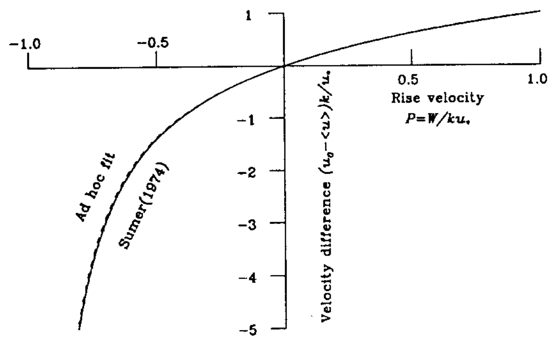

Fig. 4. Difference between the particle transport velocity u_0 and the mean water velocity $\langle u \rangle$ for rising or sinking particles in a turbulent open-channel flow.

Rising particles have effective speed slightly greater than $\langle u \rangle$ while sinking particles can travel much slower than $\langle u \rangle$.

For the shear dispersion coefficient E_0 we simply quote the formula derived by Smith [14]

$$E_0 = \frac{hu_*}{k^3} \sum_{m=1}^{\infty} \frac{(2m+1)}{m^3(m+1)^3} \prod_{j=1}^{m} \left[\frac{1 - P/j}{1 + P/j} \right] . \tag{7.6a}$$

Truncation at $m = 6$ gives four figure accuracy (uniformly over the permitted range of P). Figure 5 gives a graph (continuous curve) of the numerical factor defined by the summation and a graph of the ad hoc approximation (dashed curve)

$$\frac{E_0 k^3}{hu_*} = 0.4041 \left[\frac{1 - P}{1 + P} \right]^{1/2} . \tag{7.6b}$$

The shear is greatest near the bed. Consequently, the shear dispersion is greatest for sinking particles. Because the velocity shear is small, the scaling of the shear dispersion E_0 is not as large as the original estimate (3.10) would suggest. Equivalently, the scaling of the longitudinal length scale L is not as large as the estimate (3.11). However, E_0 is numerically a factor of forty (k^{-4}) larger than an eddy diffusivity [3].

Fischer [9] suggests that the vertical profile of the transverse eddy diffusivity $\kappa_2^{(0)}$ can be represented

$$\kappa_2^{(0)} = 0.2hu_*\eta^{1/3} . \tag{7.7}$$

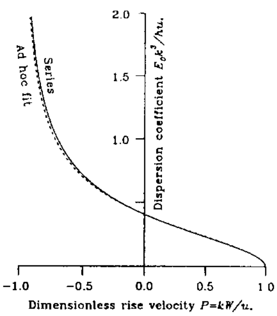

Fig. 5. Longitudinal shear dispersion coefficient E_0 for rising or sinking particles in a turbulent open-channel flow.

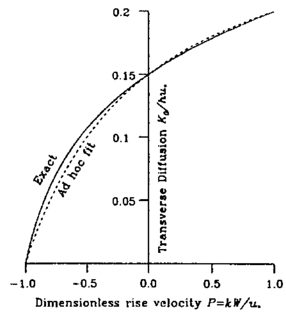

Fig. 6. Weighted average transverse turbulent diffusion K_0 for rising or sinking particles in a turbulent open-channel flow.

The corresponding vertical weighted average K_0 can be written in terms of Gamma functions:

$$K_0 = \langle \gamma \kappa_2^{(0)} \rangle = 0.15 h u_* \frac{\Gamma\left(\frac{4}{3} + P\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma(1+P)} \,. \tag{7.8a}$$

Figure 6 shows that the weak increase in turbulence away from the bed is reflected in a weak increase of K_0 with P. An ad hoc approximation to K_0 (dashed curve in Fig. 6) is

$$K_0 = 0.3 h u_* \frac{(1+P)}{2+P} \,. \tag{7.8b}$$

8. Nonlinear terms

Several of the nonlinear terms involve the turbulence reduction as characterised by the shape function $R(z)$. A commonly used (and analytically convenient) turbulence model for turbulence suppression in the proximity of boundaries was given by Monin and Obukhov [4]. The attenuation length (Monin-Obukhov length) of turbulence generated at the bed is inversely proportional to the upward diffusive flux of mass. Equivalently, the attenuation rate is directly proportional to the diffusive flux:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\nu_{13}^{(1)}/\nu_{13}^{(0)}\right) = \frac{5k\alpha g}{u_*^3}\kappa_3^{(0)}\partial_z\gamma\langle c\rangle = \frac{5k\alpha g}{u_*^3} W\gamma\langle c\rangle \,, \tag{8.1}$$

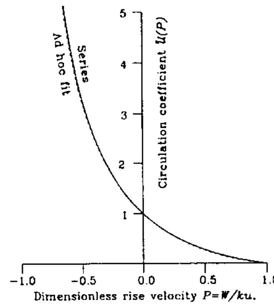

Fig. 7. Coefficient of proportionality $\hat{U}$ between the horizontal circulation of water and the stratification for a turbulent open-channel flow.

where the factor 5 is an empirical constant. The resulting formula for the shape function $R(z)$ is

$$R(z) = \frac{5k\alpha g}{u_*^3} W \int_{-h}^{z} \gamma \, \mathrm{d}z' . \tag{8.2}$$

Thus, the turbulence reduction is modelled as being proportional to the total weight (or buoyancy) of the particles from the bed up to the level z of interest. An equivalent specification of this Monin-Obukhov model in terms of the Richardson number $Ri^{(1)}$ was stated earlier in equation (5.3b). Since the nonlinear terms involve integrals of $R(z)$, there is reduced sensitivity to the particular selection of a turbulence model.

To calculate the horizontal stream function Ψ (or Φ), we first need to evaluate the weight factor $\mathfrak{U}$ as defined in the integral (5.5b). Conveniently, the Monin-Obukhov model (8.2) for the turbulence reduction leads to integrals that have already been evaluated for the linear theory [13]. The micro-hydrodynamics of rising (or sinking) particles gives the vertical Péclet number P as being proportional to the buoyancy factor α. Thus, $\mathfrak{U}$ has a second-order zero at $P = 0$. We chose to represent $\mathfrak{U}$ in a way which removes this double zero:

$$\mathfrak{U} = \frac{5k\alpha g h P}{2u_*} \hat{U} = \frac{5\alpha g h W}{2u_*^2} \hat{U} , \tag{8.3a}$$

with

$$\hat{U} = 1 + P - \frac{2k(u_0 - \langle u \rangle)}{u_*} . \tag{8.3b}$$

Figure 7 compares the exact result for $\hat{U}$ (continuous curve) with the ad hoc approximation (dashed curve)

$$\hat{U} = \left(1 - P\left[\frac{\pi^2}{3} - 2\right] + P^3\left[\frac{\pi^2}{3} - 3\right]\right) / (1 + P) . \tag{8.3c}$$

The marked growth of $\hat{U}(P)$ for negative P gives a qualitative indication how effective sinking particles are at reducing the drag, speeding up the flow and therefore inducing a horizontal circulation. By contrast, rising particles are in the wrong part of the flow to speed up the flow significantly.

We remark that for a parallel turbulent flow in water of non-uniform depth $h(y)$ the local bulk velocity $\langle u \rangle$ and the friction velocity u_* both vary as $h^{1/2}$. From the representation (8.3a)

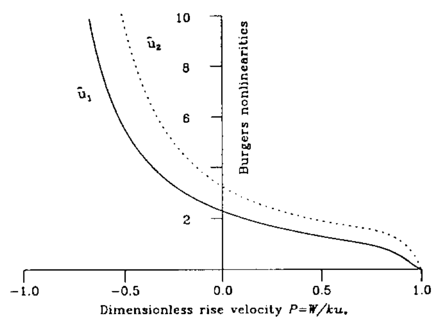

Fig. 8. Coefficients of proportionality $\hat{u}_1(\hat{u}_2)$ between the changed particle transport velocity and the local (averaged) concentration for a turbulent open-channel flow.

for $\mathfrak{U}$ we can infer that $\mathfrak{U}$ would only vary as a consequence of the changes in the vertical Péclet number P. Hence, to a good approximation the weighted average concentration $\tilde{c}$ defined in equation (6.7b) can be interpreted as being the conventional cross-sectionally averaged concentration $\bar{c}$.

For the Burgers [10] nonlinearity terms u_1, u_2 the integrable singularities (near the bed) of the double integrals (6.6a, b) can be eliminated by the use of a slightly distorted vertical coordinate. As in the representation (8.3a) for $\mathfrak{U}$, we factor out the multiple zero associated with $P = 0, \alpha = 0$:

$$u_1 = \frac{5k\alpha ghP^2}{2u_*}\hat{u}_1\,, \qquad u_2 = \frac{5k\alpha ghP^2}{2\langle u^{(0)}\rangle}\hat{u}_2\hat{U}\,. \tag{8.4}$$

As noted in Section 6, the u_2 term is anomalously small by virtue of the $\langle u^{(0)}\rangle$ denominator and can be ignored relative to the u_1 nonlinearity. It deserves comment that for rising particles (with α and P positive) the overall Burgers nonlinearities u_1, u_2 are positive while for sinking particles u_1, u_2 are negative. This is because stratification tends to speed up the flow near the free surface with a compensating slowing down of the flow near the bed. Figure 8 shows the numerical quadrature results for the dimensionless nonlinear coefficients $\hat{u}_1, \hat{u}_2$. (The product nonlinearity $\hat{u}_2\hat{U}$ varies too rapidly for convenient graphical presentation.)

The formula (6.6f) for K_1 has two contributions $K_1^{(B)}$ and $K_1^{(V)}$, associated with the buoyancy-driven and drag reduction terms in the formula (5.10a) for $v^{(0)}$. The triple integral B-term needs to be evaluated numerically. When the multiple zero at $P = \alpha = 0$ is accounted for, the (buoyancy-driven) B-contribution to K_1 has the representation

$$K_1^{(B)} = \frac{\alpha gh^2P}{ku_*}\hat{K}_B\,. \tag{8.5}$$

Once again, the necessary double integrals (5.11, 6.6f) that comprise ξ and the drag reduction contribution to K_1 have been evaluated in the linear theory:

$$\xi k\frac{\langle u^{(0)}\rangle}{u_*} = \frac{1+3P}{4} - \frac{(1+P)}{2}\frac{k(u_0 - \langle u\rangle)}{u_*}\,, \tag{8.6a}$$

$$\int_{-h}^{0}\frac{(-z)}{\nu_{23}^{(0)}}\int_{z}^{0}(\gamma - 1)\,\mathrm{d}z'\,\mathrm{d}z = \frac{h^2}{u_*^2}(u_0 - \langle u\rangle)\,. \tag{8.6b}$$

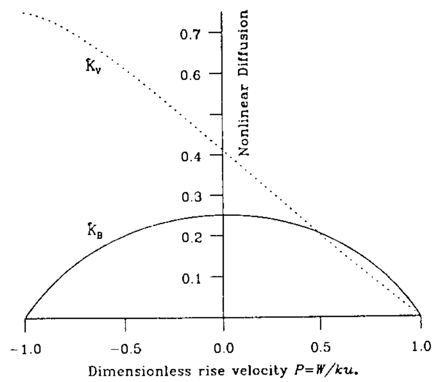

Fig. 9. Coefficients of proportionality $\hat{K}_B$, $\hat{K}_v$ between the enhanced transverse mixing and the particle concentration for a turbulent open-channel flow.

As anticipated in Section 5, the parameter ξ is anomalously small as a consequence of the relatively weak shear. The drag reduction contribution to K_1 is negative and has the representation

$$K_1^{(V)} = -\frac{\alpha g h^2 P}{k^2 \langle u^{(0)} \rangle} \hat{K}_V \,. \tag{8.7a}$$

An ad hoc approximation to the numerical factor $\hat{K}_V$ is

$$\hat{K}_V = \frac{1}{8}(1-P)\left[1-\left(\frac{\pi^2}{3}-3\right)P\right]\left[\frac{\pi^2}{3}+P\left(\frac{\pi^2}{3}-3\right)P^2\right] . \tag{8.7b}$$

Figure 9 shows the numerical results for the dimensionless coefficients $\hat{K}_B$ and $\hat{K}_V$. The comparatively large value of $\hat{K}_V$ partially compensates for the anomalously small multiplicative factor.

As P approaches -1, the negative $K_1^{(V)}$ term (8.7a) eventually dominates the positive $K_1^{(B)}$ term (8.5). Thus, the nonlinear diffusivity K_1 becomes negative. In this extreme circumstance, of dense particles which are confined closely to the channel bed, the model equation (1.1) becomes ill conditioned and numerically unstable. The implicit assumptions of slow time evolution and of smooth spatial structure cease to be valid. The manner in which the flow has become unstable (to nonlinear concentration waves) requires a different type of mathematical analysis. To avoid that unstable regime, the present analysis has to be restricted to values of P well above -1.

9. Use of the figures

From the dimensionless information conveyed in Figs. 5–9 it is straightforward to determine the dimensional coefficients in the model equation (1.1). As an illustrative example we give the flow specification

$$h = 2m, \quad u_* = 0.01\text{m}s^{-1}, \quad \langle u^{(0)} \rangle = 0.1\text{m}s^{-1}, \quad k = 0.4 \,. \tag{9.1}$$

For spherical sand particles of radius 2×10^{-5}m with density twice that of water and concentrations measured in parts per million the relevant particle specification is

$$W = -2 \times 10^{-3}\text{m}s^{-1}, \quad \alpha g = -10^{-5}\text{m}s^{-2} \,. \tag{9.2}$$

For air bubbles of the same size with a dirty surface the signs of W and αg would be reversed. The units of parts per million for concentrations are appropriate for the regime suggested by the illustrative example (3.13, 14).

The vertical Péclet number has the value

$$P = \frac{W}{ku_*} = -0.5 \tag{9.3}$$

for sand (and +0.5 for air bubbles). The marked difference between the equilibrium vertical concentration profiles γ for sand and air bubbles is shown in Fig. 3.

From Fig. 4 or the approximation (7.5) we can determine the dimensionless velocity of the sand particles relative to the water velocity (with bubble results in brackets)

$$(u_0 - \langle u \rangle)k/u_* = -1.36 \quad (0.62)\,. \tag{9.4a}$$

Hence, we can evaluate the dimensional transport velocity

$$u_0 = 0.066\mathrm{m}s^{-1} \quad (0.155\mathrm{m}s^{-1})\,. \tag{9.4b}$$

The sand near the bed is carried along much more slowly than the bubbles near the surface.

Next from Fig. 5 or the approximation (7.6b) we can determine the dimensionless shear dispersion coefficient

$$\frac{E_0 k^3}{hu_*} = 0.7 \quad (0.23)\,. \tag{9.5a}$$

The corresponding dimensional results are

$$E_0 = 0.22\mathrm{m}^2 s^{-1} \quad (0.073\mathrm{m}^2 s^{-1})\,. \tag{9.5b}$$

The sand near the bed is subject to stronger shear dispersion than the bubbles near the surface. Proceeding to Fig. 6, we evaluate the dimensionless and dimensional turbulent transverse diffusivity for the sand (and bubbles)

$$\frac{K_0}{hu_*} = 0.11 \quad (0.18)\,, \quad K_0 = 0.0022\mathrm{m}^2 s^{-1} \quad (0.0036\mathrm{m}^2 s^{-1})\,. \tag{9.6a,b}$$

The sand near the bed experiences slightly weaker horizontal eddies and turbulent mixing than the bubbles.

The new feature of this paper is the calculation of the nonlinear buoyancy/stratification effects upon the horizontal evolution. From Fig. 7 or the approximation (8.3c) we can determine the dimensionless horizontal circulation speed associated with the presence of the sand (or bubbles)

$$\hat{U} = 3.2 \quad (0.26)\,. \tag{9.7a}$$

The corresponding dimensional velocities, for concentrations measured in parts per million, are

$$\mathfrak{U} = 0.0032\mathrm{m}s^{-1}ppm^{-1} \quad (0.00026\mathrm{m}s^{-1}ppm^{-1})\,. \tag{9.7b}$$

Thus, depth averaged sand concentrations of 20 parts per million could double the water speed. Bubbles higher up in the flow do not cause quite as much speeding up.

For the Burgers nonlinearities the dimensionless results from Fig. 8 are

$$\hat{u}_1 = 5.6 \quad (1.3)\,, \quad \hat{u}_2 = 9.7 \quad (1.9)\,. \tag{9.8a,b}$$

The corresponding dimensional velocities, for particle volume fractions measured in parts per million, are

$$\begin{aligned} u_1 &= -0.0028\text{m}s^{-1}ppm^{-1} \quad (0.00065\text{m}s^{-1}ppm^{-1})\,, \\ u_2 &= -0.00016\text{m}s^{-1}ppm^{-1} \quad (2.5\times 10^{-5}\text{m}s^{-1}ppm^{-1})\,. \end{aligned} \tag{9.8c,d}$$

As noted earlier, the u_2 nonlinearity is anomalously small and could be neglected.

Except in regions where $\tilde{c}$ greatly exceed $\langle c\rangle$, the u_2 Burgers nonlinearity can be ignored relative to the u_1 term. It deserves emphasis that for dense particles the nonlinearity is negative counteracts the horizontal circulation, while and for buoyant particles the nonlinearity is positive and augments the horizontal circulation. (In both cases the flow speed near the bed is reduced with a compensating increased flow speed near the free surface.)

Finally from Fig. 9 we can determine the dimensionless nonlinear contributions to the transverse diffusion

$$\hat{K}_B = 0.2 \quad (0.2)\,, \quad \hat{H}_V = 0.62 \quad (0.2)\,. \tag{9.9a,b}$$

$$\begin{aligned} K_1^{(B)} &= 0.01\text{m}^2 s^{-1}ppm^1 \quad (0.001\text{m}^2 s^{-1}ppm^{-1})\,, \\ K_1^{(V)} &= -0.00077\text{m}^2 s^{-1}ppm^{-1} \quad (-0.00025\text{m}^2 s^{-1}ppm^{-1})\,. \end{aligned} \tag{9.9c,d}$$

When we compare of the composite diffusivities

$$K_1^{(B)} + K_1^{(V)} = 0.00023\text{m}^2 s^{-1}ppm^{-1} \quad (0.00075\text{m}^2 s^{-1}ppm^{-1}), \tag{9.9e}$$

with K_0, we find that sand (bubble) volume fractions of order 10 parts per million are required for the transverse density currents to double the effective rate of transverse mixing.

10. Concluding remarks

By construction the model equation (1.1) is complicated. The reduction from four equations (2.1a-d) in three spatial dimensions to a single equation (1.1) in two horizontal dimensions has been done with a minimum loss of physical processes. In the early states after discharge, when the particle volume fraction may be much greater than 10 parts per million, other modes of nonlinearity will be important [1]. However, as the particle patch widens the final and largest effects of buoyancy and stratification will be described by the model equation (1.1).

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